A LOWER BOUND FOR THE SCALAR CURVATURE OF NONCOMPACT NONFLAT RICCI SHRINKERS

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ABSTRACT. We show that recent work of Ni and Wilking [7] yields the result that a noncompact nonflat Ricci shrinker has at most quadratic scalar curvature decay. The examples of noncompact Kähler–Ricci shrinkers by Feldman, Ilmanen, and Knopf [5] exhibit that this result is sharp.

Let $(\mathcal{M}^n, g, f)$ be a complete shrinking gradient Ricci soliton (*Ricci shrinker* for short) with $R_{ij} + \nabla_i \nabla_j f - \frac{1}{2} g_{ij} = 0$ and $R + |\nabla f|^2 - f = 0$. Bing-Long Chen [2] proved that $R \geq 0$. If $(\mathcal{M}, g)$ is not isometric to Euclidean space, then $R > 0$ (see Stefano Pigola, Michele Rimoldi, and Alberto Setti [8] and Shijin Zhang [9]).

Recently, Lei Ni and Burkhard Wilking [7] proved that on any noncompact nonflat Ricci shrinker and for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that $R(x) \geqslant C_\epsilon d(x,O)^{-2-\epsilon}$ wherever $d(x,O)$ is sufficiently large. The purpose of this note is to observe the following version of their result.

Theorem 1. *Let $(\mathcal{M}^n, g, f)$ be a complete noncompact nonflat shrinking gradient Ricci soliton. Then for any given point $O \in \mathcal{M}$ there exists a constant $C_0 > 0$ such that $R(x) d(x,O)^2 \geqslant C_0^{-1}$ wherever $d(x,O) \geqslant C_0$. Consequently, the asymptotic scalar curvature ratio of g is positive.*

Proof. Recall that Huai-Dong Cao and De-Tang Zhou [1] proved that there exists a positive constant C_1 such that f satisfies the estimate:

$$\frac{1}{4}\left[(d(x,O) - C_1)_+\right]^2 \leq f(x) \leq \frac{1}{4}\left(d(x,O) + 2f(O)^{\frac{1}{2}}\right)^2, \tag{1}$$

where $c_+ \doteqdot \max(c, 0)$ (see also Fu-Quan Fang, Jian-Wen Man, and Zhen-Lei Zhang [4] and, for an improvement, Robert Haslhofer and Reto Müller [6]). Define the f-Laplacian $\Delta_f \doteqdot \Delta - \nabla f \cdot \nabla$. We have $0 < R + |\nabla f|^2 = f = \frac{n}{2} - \Delta_f f$. Recall that (see [3] for example)

$$\Delta_f R = -2 |\mathrm{Rc}|^2 + R. \tag{2}$$

Note that

$$\Delta_f \left(f^{-1}\right) = f^{-1} - f^{-2}\left(\frac{n}{2} - 2\frac{|\nabla f|^2}{f}\right), \tag{3}$$

$$\Delta_f \left(f^{-2}\right) = 2f^{-2} - f^{-3}\left(n - 6\frac{|\nabla f|^2}{f}\right). \tag{4}$$

Using (2) and (3), we compute for any $c > 0$

$$\Delta_f \left(R - cf^{-1}\right) \leqslant R - cf^{-1} + cf^{-2}\left(\frac{n}{2} - 2\frac{|\nabla f|^2}{f}\right). \tag{5}$$

Define $\phi \doteqdot R - cf^{-1} - cnf^{-2}$. By (4) we obtain

$$\Delta_f \phi \leqslant \phi - cnf^{-3}\left(\frac{f}{2} - n\right) - cf^{-4}\left(2f + 6n\right)\left|\nabla f\right|^2. \tag{6}$$

Choosing $c > 0$ sufficiently small, we have $\phi > 0$ inside $B(O, C_1 + 3n)$, where C_1 is as in (1). If $\inf_{\mathcal{M} - B(O, C_1+3n)} \phi \doteqdot -\delta < 0$, then by (1) there exists $\rho > C_1 + 3n$ such that $\phi > -\frac{\delta}{2}$ in $\mathcal{M} - B(O, \rho)$. Thus a negative minimum of ϕ is attained at some point x_0 outside of $B(O, C_1 + 3n)$. By the maximum principle, evaluating (6) at x_0 yields $\frac{f(x_0)}{2} - n \leq 0$. However, (1) implies that $f(x_0) \geqslant \frac{9n^2}{4}$, a contradiction. We conclude that $R \geq cf^{-1} + cnf^{-2}$ on $\mathcal{M}$. The theorem follows from (1). □

Remark. Mikhail Feldman, Tom Ilmanen, and Dan Knopf [5] constructed complete noncompact Kähler–Ricci shrinkers on the total spaces of k-th powers of tautological line bundles over the complex projective space $\mathbb{CP}^{n-1}$ for $0 < k < n$. These examples, which have Euclidean volume growth and quadratic scalar curvature decay, show that Theorem 1 is sharp.

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